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W. L. Black

Design of Electromagnetic Torque Rods

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LINCOLN LABORATORY

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Group 63

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ABSTRACT

This report is a compilation of considerations involved in the design of electro-magnetic rods for satellite erection. Formulae are derived showing the effect of various choices of core and wire material and weight, number of cores, length of cores, number of turns per core, and wire area on the magnetic moment, hysteresis power, hold power, hold voltage, and switching time. These formulae are then applied to the problem of producing a required magnetic moment at the least cost in weight and power consumption.

Accepted for the Air Force
Franklin C. Hudson
Chief, Lincoln Laboratory Office

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1. INTRODUCTION

This report is a compilation of considerations involved in the design of electro-magnetic rods for satellite erection. Formulae are derived showing the effect of various choices of core and wire material and weight, number of cores, length of cores, number of turns per core, and wire area on the magnetic moment, hysteresis power, hold power, hold voltage, and switching time. These formulae are then applied to the problem of producing a required magnetic moment at the least cost in weight and power consumption.

2. DERIVATION OF CORE AND WINDING RELATIONS

The following sections are devoted to the derivation of those equations that describe the functioning of a single core with a single winding. Subsequent sections will treat questions of multiple rods and/or windings.

2.1 Determination of Fields in Core

In this section we derive the dependence of the core fields on the drive current, etc. This simple derivation makes certain gross approximations, but is justified by the more complete analysis in Appendix 1. The core geometry and relevant notation are introduced in Figure 1.

We begin by assuming a uniform axial magnetization, M. This magnetization charges the rod ends to a surface density of $\mu_o M$. We shall treat the end charges as points with total charge $\mu_o MA$ and compute the resulting demagnetizing H along the Z axis. It is easily verified that

$$H_d(z) = \frac{2\,MA}{\pi\,\ell^2} \quad \frac{1 + (2z/\ell)^2}{\left[1-(2z/\ell)^2\right]^2}$$

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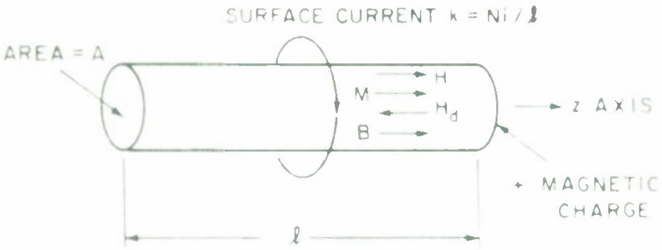

H = TOTAL H IN CORE

H_d = DEMAGNETIZING H DUE TO MAGNETIC CHARGE

M = CORE MAGNETIZATION

B = CORE B

f = NUMERICAL CONSTANT

CV = Aℓ = CORE VOLUME

N = TURNS/WINDING

i = CURRENT IN WINDING

Figure 1 Core geometry and notation

H_d is clearly non-uniform; it is a minimum at the center ($z = 0$) and tends to infinity at the ends ($z = \pm \ell/2$) of the rod. It is convenient to introduce an "average" H_d and employ it as though the demagnetization field were uniform. Hence, we shall supplant the previous relation by

$$H_d = \frac{f\,MA}{\ell^2} = \frac{f\,M\,CV}{\ell^3}$$

The numerical factor f must be determined experimentally or by a more intricate analysis such as that in Appendix 1. It generally lies between three and ten.

The applied current produces a nearly uniform field in the core region of magnitude k. The total H field is

$$H = k - H_d = k - f\,M\,\frac{CV}{\ell^3}$$

We can eliminate M from this expression by employing

$$\frac{B}{\mu_o} = H + M$$

This gives the load-line equation

$$\left(1 - \frac{f\,CV}{\ell^3}\right) H + \frac{f\,CV}{\ell^3}\,\frac{B}{\mu_o} = k$$

Since we are interested only in long thin rods

$$f\,\frac{CV}{\ell^3} < \frac{1}{1000}$$

and so this ratio may be ignored in the coefficient of H.

The load-line can be combined with the B-H curve for the material to obtain the operating point of the material. Figure 2 shows this construction.

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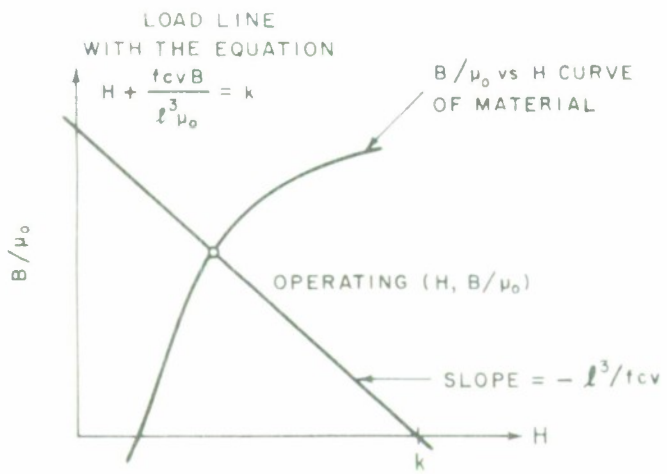

Figure 2 Construction for determining operating point at core.

2.2 Determination of the Magnetic Moment of the Saturated Rod

Outside the core, the effect of the current sheet is identical to that of a volume dipole moment of density k. Hence the magnetic moment $\mathfrak{M}$ is

$$\mathfrak{M} = CV\ (k + M)$$

We already have expressions for k and M in terms of B and H. Inserting these gives

$$\mathfrak{M} = CV\left[\left(1 + f\frac{CV}{\ell^3}\right)\frac{B}{\mu_o} - f\frac{CV}{\ell^3}H\right]$$

For saturated, high permeability, long, thin rods we approximate

$$\mathfrak{M} = CV\frac{B_s}{\mu_o}$$

where B_s is the saturation B-field of the core.

2.3 Determination of the Currents Required for Saturation and Holding

The required saturation and holding currents depend on the core material. Figure 3 shows a typical B-H curve and introduces some notation. From Figure 2

$$k = \frac{Ni}{\ell} = H + f\frac{CV}{\ell^3}\frac{B}{\mu_o}$$

Hence: i_s = saturation current = $k_2\ell/N$

i_h = hold current = $k_3\ell/N$

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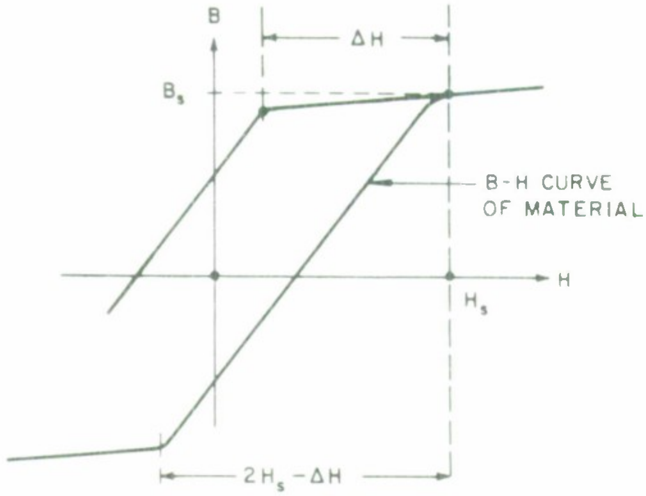

H_s = H REQUIRED FOR SATURATION

B_s = SATURATION B

ΔH = "WIDTH" OF LOOP

H_h = H USED FOR HOLDING. H_h MUST AT LEAST EQUAL $H_s - \Delta H$

Figure 3 A typical B-H curve and some relevant parameters.

where we have defined the two parameters

$$k_2 = H_s + f\mathfrak{M}/\ell^3$$

$$k_3 = H_h + f\mathfrak{M}/\ell^3$$

2.4 Determination of Hysteresis Loss in Core

The hysteresis loss/cycle is

$$\text{hyst. loss/cycle} = 2\,B_s\,\Delta H\,CV = 2\,\mu_o\mathfrak{M}\,\Delta H$$

Hence, if the length of a cycle is T, the hysteresis power dissipation is

$$P_{hyst} = \frac{2\mu_o\mathfrak{M}\,\Delta H}{T}$$

2.5 Determination of Switching Inductance

The switching inductance is computed by dividing the flux linkage change by the current change during traversal of the non-saturated portion of the B-H curve. From Figure 3, the change during switching in B and H are

$$\text{change in B} = 2\,B_s$$

$$\text{change in H} = 2\,H_s - \Delta H$$

Letting N equal the number of turns, we see

$$\text{flux linkage change} = 2\,AB_sN = 2\,\mu_o\mathfrak{M}\,N/\ell$$

Since we have an expression for current in terms of B and H we can compute the current change:

$$\text{current change} = \frac{\ell}{N}\left[2 H_s - \Delta H + \frac{f\,CV}{\ell^3} \cdot \frac{2 B_s}{\mu_o}\right]$$

$$= \frac{2\ell}{N}\left[H_s - \frac{\Delta H}{2} + \frac{f\,\mathfrak{M}}{\ell^3}\right]$$

Hence the switching inductance L is

$$L = \mu_o \mathfrak{M}\, N^2/\ell^2 \left(k_2 - \frac{\Delta H}{2}\right)$$

2.6 Wire Area and Coil Resistance

The winding geometry and notation are shown in Figure 4. We shall express the resistance of the coil in terms of WV, CV, N, ℓ and ρ. From the figure

$$\xi\, WV = \pi\, \ell\, R^2 - CV$$

Hence

$$R = \left(\frac{CV + \xi\, WV}{\pi\, \ell}\right)^{1/2}$$

But

$$r = \left(\frac{CV}{\pi\, \ell}\right)^{1/2}$$

The mean turn length is $\pi\,(R + r)$ so the wire area is:

$$A_{wire} = \frac{WV}{N\pi\,(R + r)} = \left(\frac{WV\,\ell}{\xi}\right)^{1/2} \Big/ N\left(\frac{\pi\, CV}{\xi\, WV}\right)^{1/2}\left[1 + \left(1 + \frac{\xi\, WV}{CV}\right)^{1/2}\right]$$

It is convenient to introduce a parameter k_1 defined by

$$k_1 = \left(\frac{\pi\, CV}{\xi\, WV}\right)\left[1 + \left(1 + \frac{\xi\, WV}{CV}\right)^{1/2}\right]^2$$

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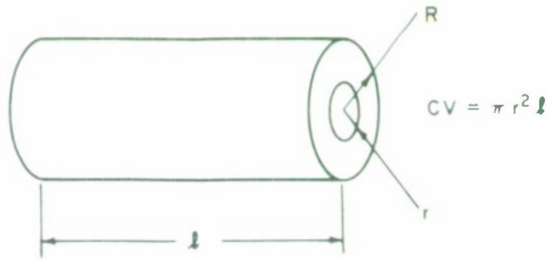

WV = CONDUCTOR VOLUME

ξ = PACKING FACTOR – THE RATIO OF WINDING VOLUME TO CONDUCTOR VOLUME

N = NUMBER TURNS

ρ = CONDUCTOR RESISTIVITY

k_1 = A DEFINED PARAMETER

Figure 4 Winding geometry and Notation.

In terms of this parameter we may write

$$A_{wire} = \left(\frac{WV \ell}{\xi k_1} \right)^{1/2} \cdot \frac{1}{N}$$

The total length of the wire is WV/A_{wire} so the resistance of the coil is

$$R_{coil} = \rho \frac{WV}{A_{wire}^2} = \rho \frac{\xi k_1 N^2}{\ell}$$

2.7 Hold Voltage and Power

The hold voltage and power are easily expressed in terms of the other parameters. The hold voltage is

$$V_{hold} = i_h R_{coil} = \rho \xi k_1 k_3 N$$

The hold power is

$$P_{hold} = i_h^2 R_{coil} = \rho \xi k_1 k_3^2 \ell$$

For future use we note the identity

$$A_{wire} V_{hold} = \left(WV \rho P_{hold} \right)^{1/2}$$

2.8 Voltage Source Switching Time Constant

The time constant appropriate for switching from an ideal voltage source is

$$\tau_{switch} = \frac{L}{R} = \frac{\mu_o (m)}{\rho \xi k_1 \ell \left(k_2 - \frac{\Delta H}{2} \right)}$$

2.9 Single Rod Formulae - Summary

$$\mathfrak{M} = CVB_s/\mu_o$$

$$R_{coil} = \rho\xi k_1 N^2/\ell$$

$$\text{where } k_1 = \left(\frac{\pi CV}{\xi WV}\right)\left[1 + \left(1 + \frac{\xi WV}{CV}\right)^{1/2}\right]^2$$

$$i_s = k_2\ \ell/N$$

$$\text{where } k_2 = H_s + f\mathfrak{M}/\ell^3$$

$$i_h = k_3\ \ell/N$$

$$\text{where } k_3 = H_h + f\mathfrak{M}/\ell^3$$

$$V_{hold} = \rho\xi\ k_1 k_3 N$$

$$P_{hyst} = 2\mu_o\mathfrak{M}\Delta H/T$$

$$P_{hold} = \rho\xi k_1 k_3^2\ \ell$$

$$A_{wire}\ V_{hold} \equiv (WV\rho\ P_{hold})^{1/2}$$

$$L = \mu_o\mathfrak{M}N^2/\ell^2\ (k_2 - \Delta H/2)$$

$$T_{switch} = \mu_o\mathfrak{M}/\rho\xi k_1\ \ell\ (k_2 - \Delta H/2)$$

3. TRADE OFFS AND OPTIMIZATIONS

This section applies the formulae derived in Section 2 to the problem of design optimization. The parameters that the designer can vary are:

Magnetic material

Winding material

Core volume

Wire volume

Core length

Wire area

Number of turns

Hold voltage

Not all of these are independent, and certain of them may be fixed by external constraints. These parameters must all be selected to produce a given $\mathcal{M}$ which is generally determined by the dynamics of the system. Moreover, the designer wants:

Minimum weight

Minimum hysteresis power loss

Minimum hold power loss

Minimum switching time

He is limited by the practical constraints that:

Wire size must not be too small

Core length must not exceed the satellite diameter

3.1 Determination of Core Material

Once $\mathfrak{M}$ has been determined and a material selected, the hysteresis power loss is fixed independent of all other design choices. The material with lowest ΔH also has the lowest hysteresis loss. However, the hysteresis loss is often a small fraction of the hold power, and the dependence of hold power on core material is somewhat more complex. The choice of material affects the hold power two ways:

(i) Materials with high B_s require less material to produce a given $\mathfrak{M}$. Thus, for a fixed system weight, more wire can be used, reducing k_1.

(ii) Materials with a low H_h will have low values of k_3.

Clearly, if we could find a material that has highest B_s and lowest H_h, it would be the best choice. Since so few materials are even contenders, the simplest approach is to work through the design procedure to be given later for each and select the one that best meets the mission requirements as to weight and hold power.

3.2 Selecting Core Length to Minimize P_{hold}

As we have seen in Section 2, the hold power is proportional to

$$P_{hold} \sim \left(1 + \frac{\beta^3}{\ell^3}\right)^2 \ell$$

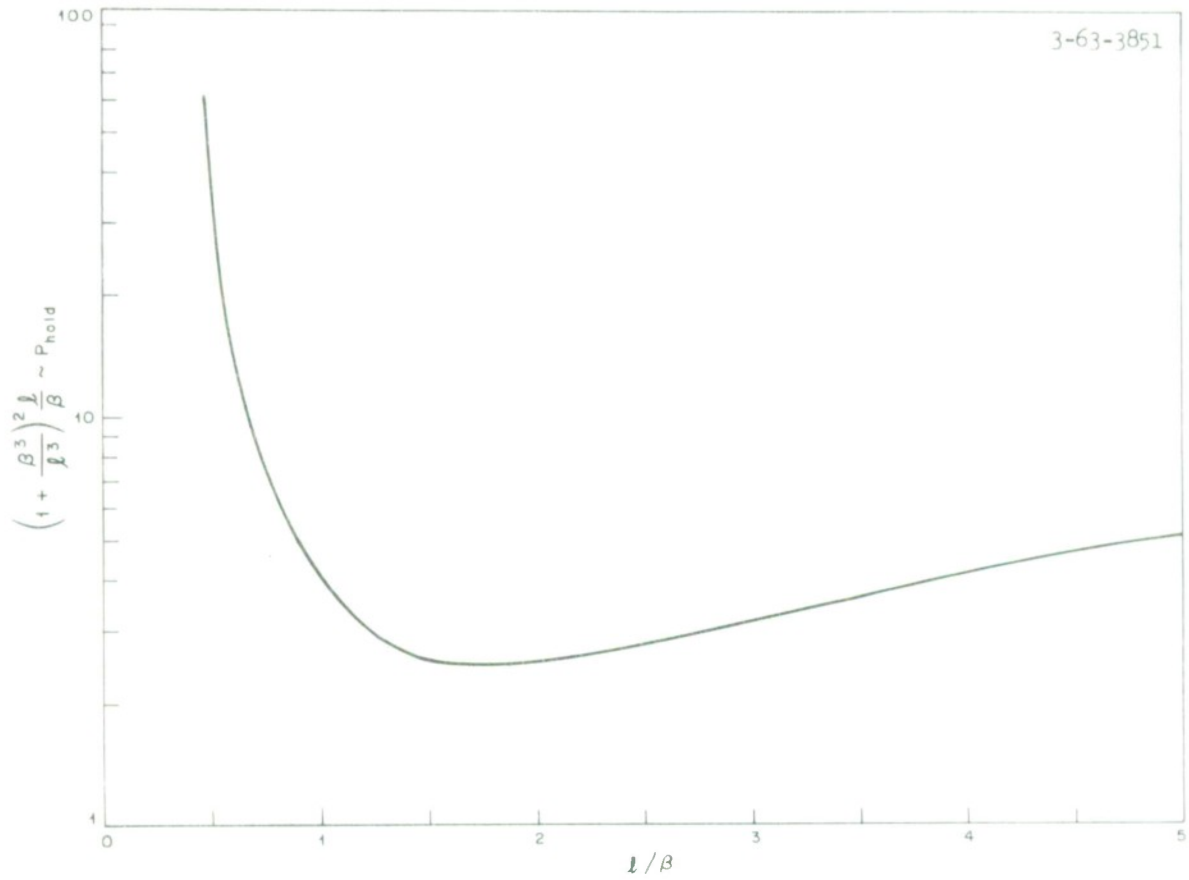

Figure 5 Graph showing how Pnold depends on core length.

where we have just defined

$$\beta^3 = f\, I\eta / H_h$$

Differentiating this with respect to ℓ gives

$$\frac{d\, P_{hold}}{d\, \ell} \sim \left(1 + \frac{\beta^3}{\ell^3}\right)\left(1 - \frac{5\,\beta^3}{\ell^3}\right)$$

Thus, as ℓ increases from zero, P_{hold} decreases to a minimum and then rises steadily. This minimum value obtains when

$$\ell = 5^{1/3}\,\beta$$

If the rod length is constrained to be less than some ℓ_{max} we should select

$$\ell = \inf\left(5^{1/3}\,\beta,\ \ell_{max}\right)$$

in order to minimize the holding power. Figure 5 displays the dependence of the holding power on core length.

3.3 Effect of Replacing a Single Core by Several Small Ones

Under certain circumstances it is advantageous to replace a single rod by several smaller ones. The total core columе must remain the same in order to produce the required magnetic moment. Moreover, if the weight is fixed the ratio WV/CV for each core is unchanged. However, the terms $k_3^2\ \ell$ do change, and a significant reduction in hold power may result. Suppose, for example, we choose to employ e rods. Then from Section 2, the total hold power is proportional to:

$$P_{hold\,-\,total} \sim \frac{e}{\varepsilon}\left[1 + \frac{\varepsilon\, \ell^3_{max}}{e\, \ell^3}\right]^2 \frac{\ell}{\ell_{max}}$$

where we have defined:

$$\varepsilon = f\mathfrak{M}/H_h\, \ell^3_{max} = \beta^3/\ell^3_{max}$$

and $\mathfrak{M}$ is the total magnetic moment of the system. We have seen in the previous section that the core length should be selected as

$$\ell = \inf\left[\left(\frac{5\varepsilon}{e}\right)^{1/3} \ell_{max}, \ \ell_{max}\right]$$

For $\frac{e}{\varepsilon} > 5$ we have $\ell < \ell_{max}$ and:

$$P_{hold\text{-}total} \sim \left(\frac{6}{5}\right)^2 5^{1/3}\left(\frac{e}{\varepsilon}\right)^{2/3}$$

When $\frac{e}{\varepsilon} < 5$ we see $\ell = \ell_{max}$ and:

$$P_{hold\text{-}total} \sim \left(\frac{e}{\varepsilon} + 2 + \frac{\varepsilon}{e}\right)$$

This latter has a minimum at $\frac{e}{\varepsilon} = 1$. Figure 6 shows how the hold power depends on the ratio e/ε. The best choice of e would be $e = \varepsilon$ but this may not be an integer. We see immediately that splitting is never profitable if $\varepsilon < 1$. If splitting is profitable, it should never proceed past $e = \varepsilon + 1 \leq 2\varepsilon < 5\varepsilon$. Hence, we need only consider the portion of the curve in Figure 6 corresponding to $\ell = \ell_{max}$.

We note that e + 1 rods require less hold power than e if and only if:

$$\varepsilon > 1$$

$$e < \varepsilon$$

$$\frac{e+1}{\varepsilon} + 2 + \frac{\varepsilon}{e+1} < \frac{e}{\varepsilon} + 2 + \frac{\varepsilon}{e}$$

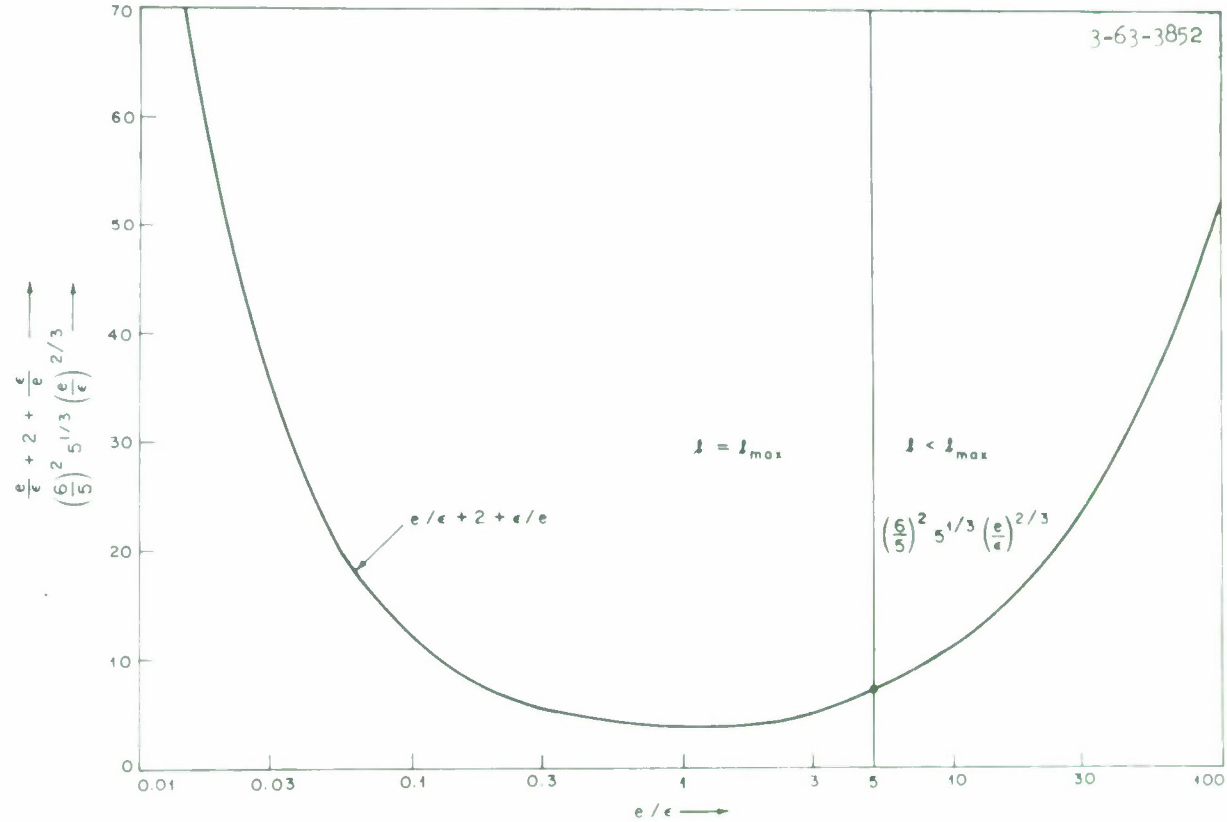

Figure 6 Graph showing how total hold power depends on number of cores used to produce a given magnetic moment.

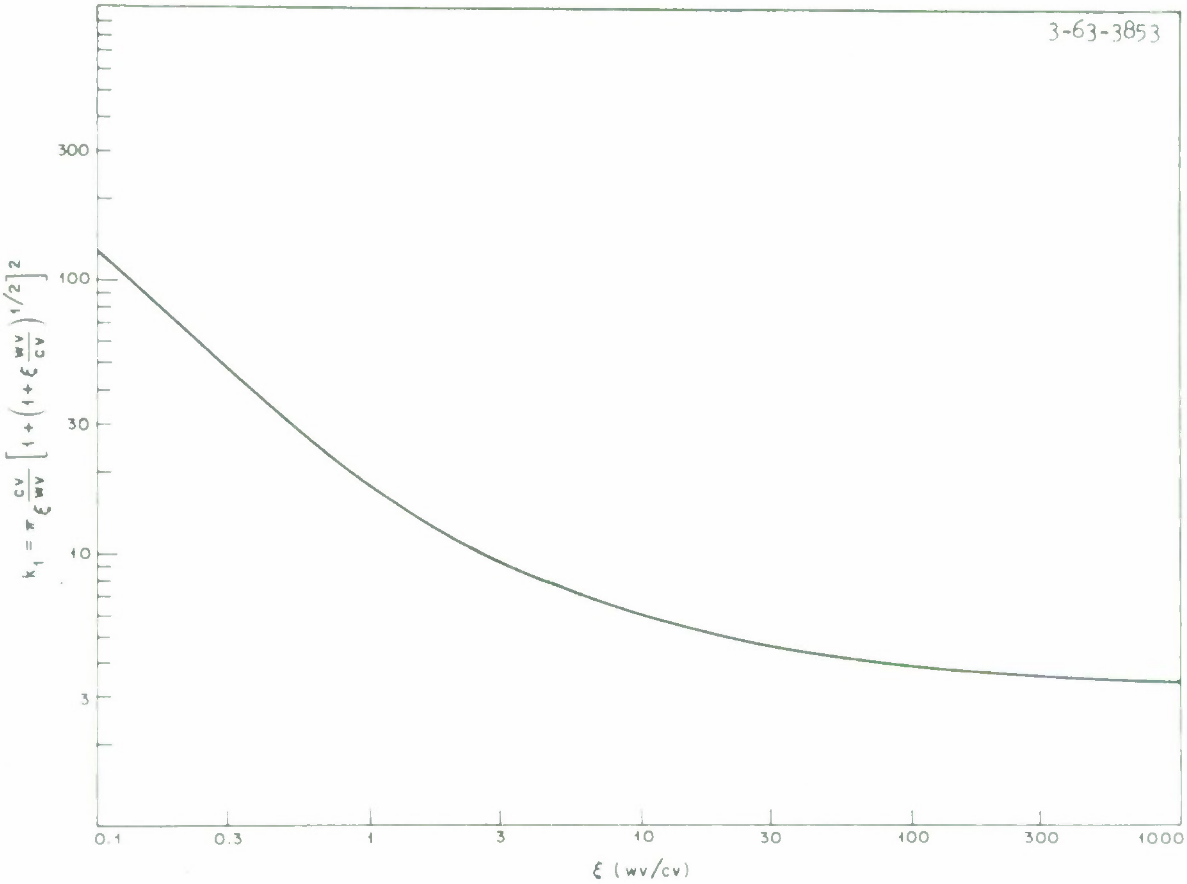

Figure 7 Graph showing the dependence of hold power on wire volume.

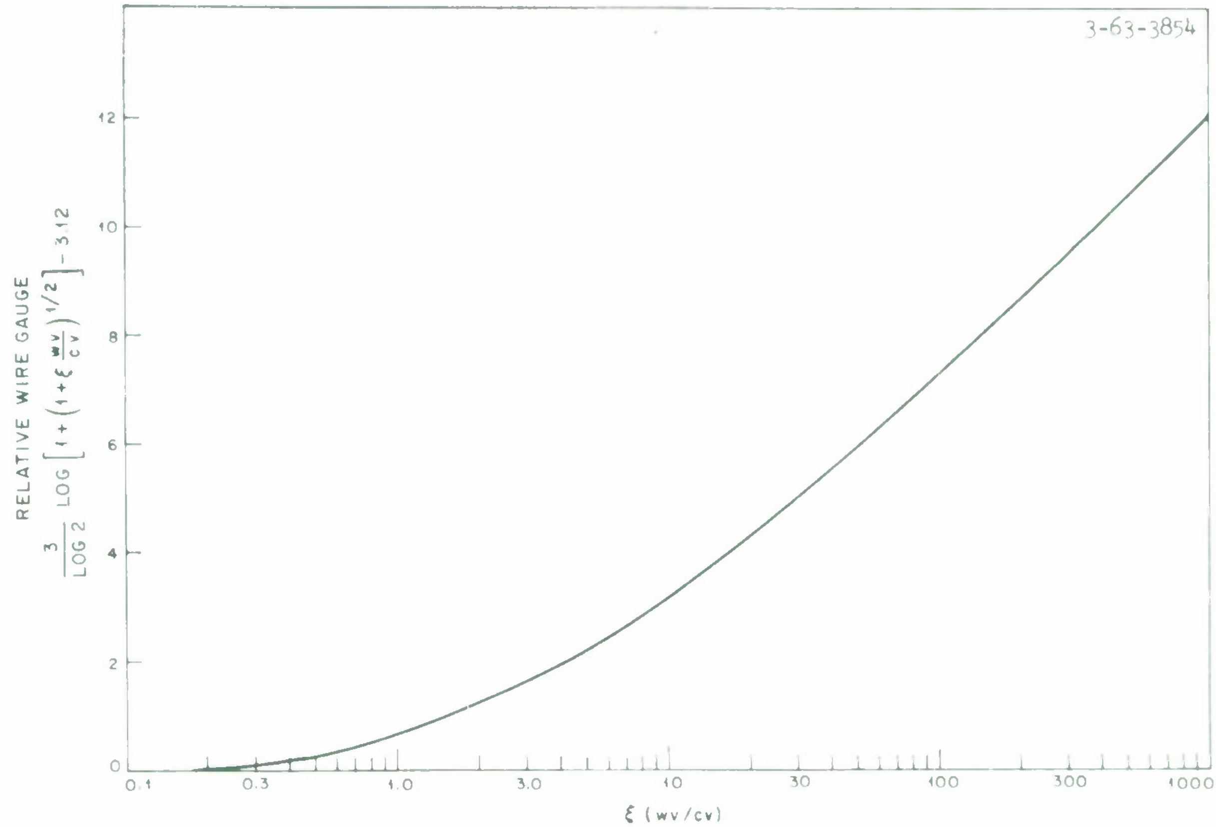

Figure 8 Graph showing how wire gauge depends on wire volume for constant hold voltage, the effect of varying wv is to change the true wire gauge by the change in relative wire gauge determined from this graph.

These are all equivalent to the statement:

$$e\,(e+1) < \varepsilon^2$$

Hence, we have the following role to determine the optimum integral number of rods:

$$e_{opt} = \begin{cases} 1 & \text{if } \varepsilon \le \sqrt{1.2} \\ 2 & \text{if } \sqrt{1.2} < \varepsilon \le \sqrt{2.3} \\ 3 & \text{if } \sqrt{2.3} < \varepsilon \le \sqrt{3.4} \end{cases}$$

etc.

For any choice of e between 1 and e_{opt} the core lengths are $\ell = \ell_{max}$. So long as e is smaller than 5ε we have the convenient formula:

$$\frac{P_{\text{hold-total-e cores}}}{P_{\text{hold-single equivalent core}}} = \frac{\frac{e}{\varepsilon} + 2 + \frac{\varepsilon}{e}}{\frac{1}{\varepsilon} + 2 + \varepsilon}$$

3.4 Wire Volume Considerations

In Section 2 we saw that the hold power was proportional to k_1. Figure 7 shows the dependence of k_1 on the ratio ξ WV/CV. The asymptote as WV $\to \infty$ is at π. From this graph we see that there is little point to using more wire volume than 10 CV/ξ, unless power is at an extreme premium. (With aluminum wire and a ferromagnetic core this corresponds to a wire to core weight ratio of approximately 2.4). However, it is true that more wire always reduces the hold power.

It is also true that increasing the wire volume may ease the winding problem by permitting larger wire area for a given hold voltage. Figure 8 shows the dependence of wire area times hold voltage on wire volume. It is based on the result, easily derivable from the formulae in Section 2, that:

$$A_{wire} V_{hold} \sim 1 + \left(\frac{1 + \xi WV}{CV} \right)^{1/2}$$

and the wire gauge formula:

$$\text{Gauge 2} - \text{Gauge 1} = \frac{3}{\text{Log 2}} \text{ Log} \left(\frac{\text{Area 1}}{\text{Area 2}} \right)$$

3.5 Winding Material for Minimum P_{hold}

In Section 2 we saw that the hold power was proportional to:

$$P_{hold} \sim \rho k_1$$

There are only two contenders for the wire material - copper and aluminum. Their densities and resistivities are shown below:

$$\rho\ (\text{Al}) = 2.83 \cdot 10^{-8}\ \text{r-m} \qquad d\ (\text{Al}) = 2.7 \cdot 10^{3}\ \text{kg/m}^{3}$$

$$\rho\ (\text{Cu}) = 1.72 \cdot 10^{-8}\ \text{r-m} \qquad d\ (\text{Cu}) = 8.9 \cdot 10^{3}\ \text{kg/m}^{3}$$

As is shown in Figure 7, for large amounts of wire k_1 approaches asymptotically to π. Under these conditions the better conductor, copper, minimizes the hold power. On the other hand for low permissible wire weights, aluminum minimizes the hold power since the increase in wire cross-section area more than compensates for the increase in resistivity. We shall assume a ferromagnetic core with a density of $8.13 \cdot 10^{3}$ kg/m^3. We shall let:

$$\eta = \text{ratio of wire weight to core weight}$$

Then

$$\frac{WV}{CV} = \frac{\text{wire weight}}{\text{wire density}} \cdot \frac{\text{core density}}{\text{core weight}}$$

$$= \eta \cdot \frac{8.13 \cdot 10^{3}}{\text{wire density}}$$

Hence, the ratio of the hold powers with aluminum and copper wire is:

$$\frac{P_{hold}(Al)}{P_{hold}(Cu)} = \frac{1}{2}\left[\frac{(1 + 3.01\,\xi\eta)^{1/2} + 1}{(1 + .914\,\xi\eta)^{1/2} + 1}\right]^2$$

The cross over point where Al and Cu yield equal hold powers occurs when:

$$\sqrt{2} = \frac{1 + (1 + 3.01\,\xi\eta)^{1/2}}{1 + (1 + .914\,\xi\eta)^{1/2}}$$

This occurs when $\xi\eta = 2.83$. Hence for $\xi\eta < 2.83$ aluminum is the better choice. We shall see that aluminum also permits the use of larger wire. Thus, it may be necessary to use aluminum even when copper would give a slightly lower hold power. Even when $\xi\eta = 102$, the hold power for aluminum exceeds that of copper by only 50 per cent. In no case is aluminum worse than copper by more than 65 per cent.

3.6 Selection of Wire Material for Maximum Wire Area at a Given Hold Voltage

Using results derived in Section 2 we see that:

$$\frac{A_{wire}(Al)}{A_{wire}(Cu)} = \left[\frac{\rho(Al)\,P_{hold}(Al)/d(Al)}{\rho(Cu)\,P_{hold}(Cu)/d(Cu)}\right]^{1/2}$$

$$= 2.33\left[\frac{P_{hold}(Al)}{P_{hold}(Cu)}\right]^{1/2}$$

As seen in the previous section, the ratio of the two hold powers always lies between .5 for $\xi\eta = 0$ and 1.65 for $\xi\eta = \infty$. Hence, the area ratio always lies between 1.65 and 2.99. Thus aluminum always permits the use of larger wire. The difference in wire gauge numbers ranges from 2.17 to 4.75.

3.7 Voltage Source Switching

One way to reverse the current in a rod is to apply a back voltage to the entire coil as shown in Figure 9. Zero time is required for the operating point to move along the saturated portions of the λ - i curve, so switching actually begins at a current of $-i_s + \Delta i$. Δi is the width of the λ - i curve -- if, as we shall assume, i_h, is selected to hold on the knee, then $\Delta i = i_s - i_h$. In any event, $\Delta i = \ell \, \Delta H/N$.

Figure 10 shows the switching transient. From it we can compute:

$$\frac{V_s}{V_h} = \frac{e^{-t/\tau} + \frac{i_s}{i_h}}{1 - e^{-t/\tau}}$$

Evidently the switch voltage must exceed the hold voltage. By direct integration one may verify that the total charge drawn from the source is:

$$Q = i_h t \left[\frac{V_s}{V_h} - \frac{\tau}{t}\left(\frac{i_s}{i_h} + 1\right)\right]$$

which can be written

$$\frac{Q}{i_h t} = (r + 1)\left[\frac{x - (1 - e^{-x})}{x\,(1 - e^{-x})}\right] - 1$$

where

$$x = t/\tau$$

$$r = i_s/i_h$$

For $x \leq 2$ the quantity in brackets is closely (within a few per cent) approximated by $.5 + .08x$. Hence, for $\frac{t}{\tau} \leq 2$ the charge drawn from the switching supply is:

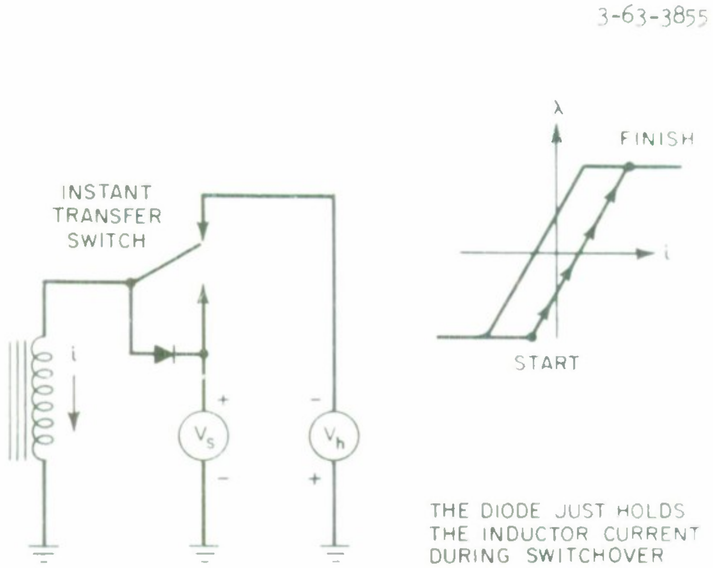

Figure 9 Voltage source switching

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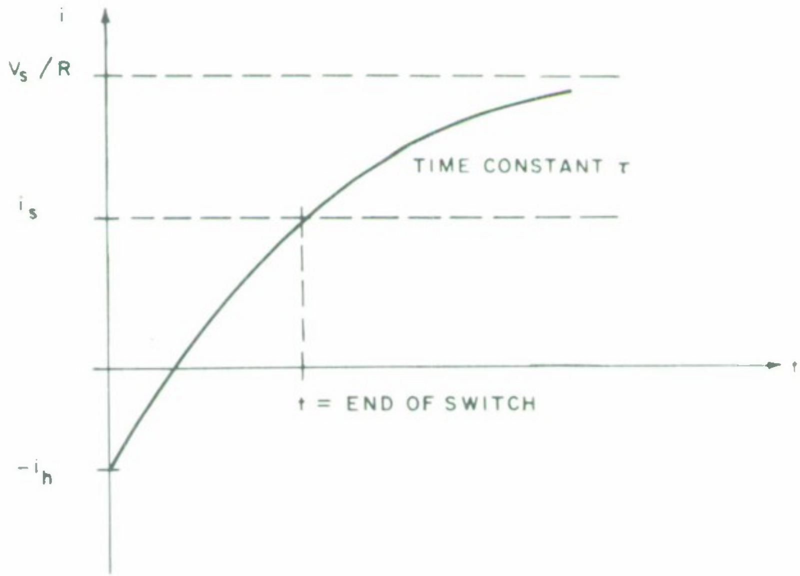

Figure 10 Corrent vs time during voltage source switching.

$$\frac{Q}{i_h t} = \left(\frac{i_s - i_h}{i_h}\right)\left(.5 + .08 \frac{t}{T}\right) + .08 \frac{t}{T}$$

The energy drawn from the supply is of course $V_s Q$. Note that when the switching source is supplying power, the hold source is not so that this drain does not represent additional power.

3.8 Summary of Design Formulae and Results

$$\ell = \inf\left(5^{1/3} \beta, \ \ell_{max}\right)$$

where $\beta^3 = f \ /H_h$

$$\begin{aligned} e_{opt} &= 1 \text{ if } \epsilon \le \sqrt{1.2} \\ &= 2 \ \sqrt{1.2} \text{ L } \epsilon \le \sqrt{2.3} \\ &= 3 \ \sqrt{2.3} \text{ L } \epsilon \le \sqrt{3.4} \qquad \text{etc.} \end{aligned}$$

where $\epsilon = \beta^3 / \ell^3_{max}$

$$\frac{P_{\text{hold-e cores}}}{P_{\text{hold-1 core}}} = \frac{e/\epsilon + 2 + \epsilon/e}{1/\epsilon + 2 + \epsilon}$$

Al gives a lower hold power than Cu for low wire weights. The cross-over occurs when

$$\xi \cdot \frac{\text{wire weight}}{\text{core weight}} = 2.83$$

Al always leads to larger wire size for fixed hold voltage. The difference is on the order of 3 gauge units.

$$\frac{V_s}{V_h} = \frac{e^{-t/\tau} + i_s/i_h}{1 - e^{-t/\tau}}$$

$$\frac{Q}{i_h t} = \frac{V_s}{V_h} - \frac{\tau}{t}\left(\frac{i_s}{i_h} + 1\right)$$

$$\simeq \left(\frac{i_s - i_h}{i_h}\right)\left(.5 + .08\ \frac{t}{\tau}\right) + .08\ \frac{t}{\tau}$$

for $\frac{t}{\tau} \leq 2.$

APPENDIX A

The Wound Prolate Ellipsoid

1. A Prolate Ellipsoid Uniformly Magnetized Along Its Axis of Rotational Symmetry

In this section we shall determine the field resulting from an ellipsoid with uniform magnetization along its rotary symmetry axis. Figure A-1 shows the geometry

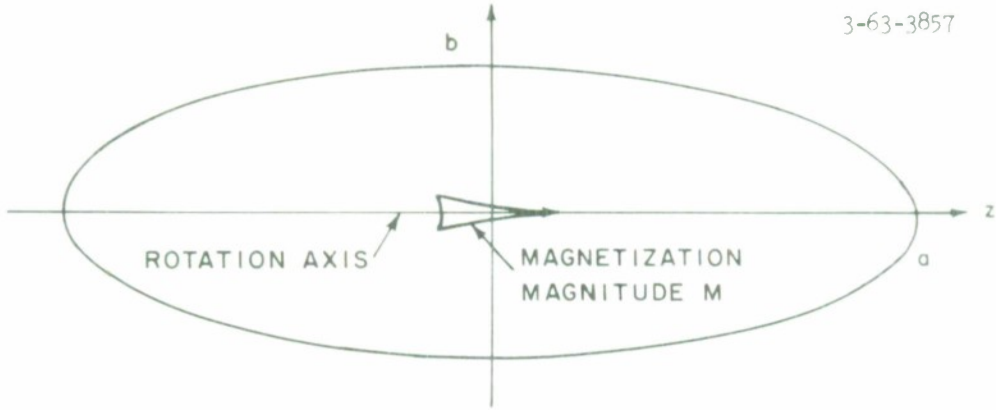

Figure A-1 The Prolate Ellispoid

The effect of the uniform magnetization is to produce a surface charge on the ellipsoid with density

$$\sigma = \mu_o M \, 1^z \cdot 1^n$$

One can easily obtain an explicit form for σ and verify the following interesting fact: if the charge is all projected normally onto the z-axis, the resulting line charge density is proportional to z -- and the total positive

charge equals $\mu_o M$ times the area of the ellipsoid at its equator. Figure A-2 shows this line charge ζ.

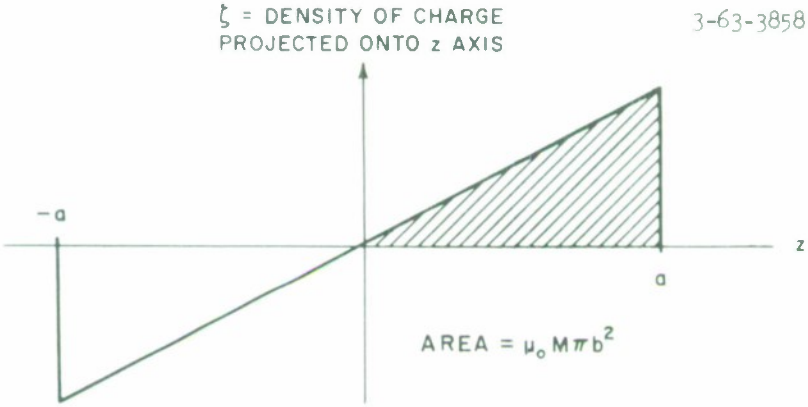

Figure A-2 The Projected Charge Density

Given the charge one can find the fields directly by Greens-function methods, or else one can solve for the magnetic potential by inserting the charge as a boundary condition and expanding in ellipsoidal harmonics. The following two facts emerge: first, the internal H-field is uniform; and second, the external field is the same as that produced by a line of charge, linearly graded, extending between the two foci, with dipole moment equal to M times the volume of the ellipsoid. Figure A-3 shows this equivalent density

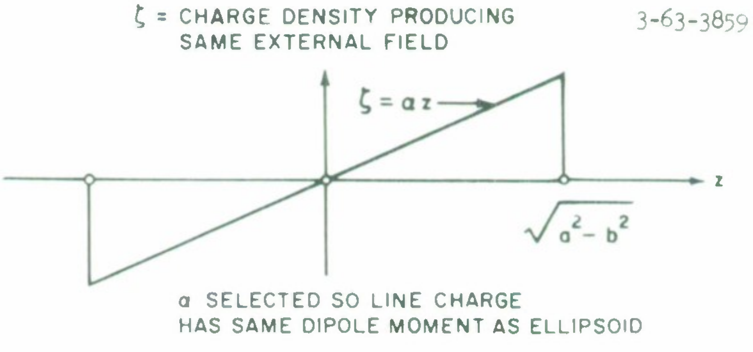

Figure A-3 Charge Producing Same External Field as Ellipsoid

We can determine α as follows:

$$\frac{2}{\mu_o} \int_0^{\sqrt{a^2 - b^2}} z \cdot \alpha z dz = \frac{4\pi}{3} a b^2 M$$

Performing the integration and solving for α gives:

$$\alpha = 2\pi \mu_o M \frac{a b^2}{(a^2 - b^2)^{3/2}}$$

Now that α has been determined we can compute the external H-field at the equator. Figure A-4 shows the relevant geometry.

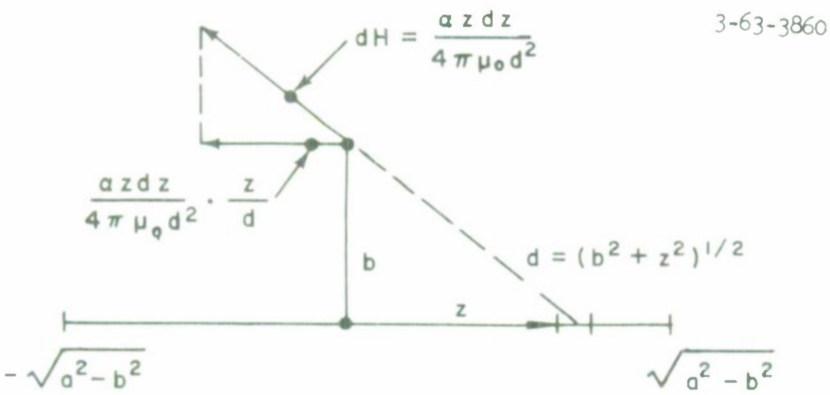

Figure A-4 The Determination of H at the Equator of the Ellipsoid

By symmetry it is evident that H will have no normal or rotary component on the equator. From the figure, the value of the H-field in the negative z direction is:

$$|H| = \int_{-\sqrt{a^2 - b^2}}^{\sqrt{a^2 + b^2}} \frac{\alpha}{4\pi\mu_o} \frac{z^2}{d^3} dz$$

$$= \frac{\alpha}{4\pi\mu_o}\left(\operatorname{Log}\frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}} - 2\frac{\sqrt{a^2 - b^2}}{a}\right)$$

Since the tangential component of H is continuous across the boundary, the uniform H field within the ellipsoid points in the negative z direction and has magnitude equal to the above. Thus the internal H opposes M and the magnitudes are related by:

$$|H| = \beta M$$

where

$$\beta = \frac{ab^2}{2(a^2 - b^2)^{3/2}}\left[\operatorname{Log}\frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}} - \frac{2\sqrt{a^2 - b^2}}{a}\right]$$

If we let $\Lambda = a/b$ we get:

$$\beta = \frac{1}{\Lambda^2 - 1}\left[\frac{\Lambda}{\sqrt{\Lambda^2 - 1}} \operatorname{Log}\left(\Lambda + \sqrt{\Lambda^2 - 1}\right) - 1\right]$$

2. Fields Due to Current Circulating About a Prolate Ellipsoid

To find the fields we exploit the equivalence of circulating current and dipole sheet. The H field due to current filament i equals (except on the sheet) that due to a sheet of dipoles with surface density i that spans the

filament. As shown in Figure A-5 a surface current whose projection on the z-axis is uniform density k, produces the same external field as an ellipsoid with uniform magnetization k.

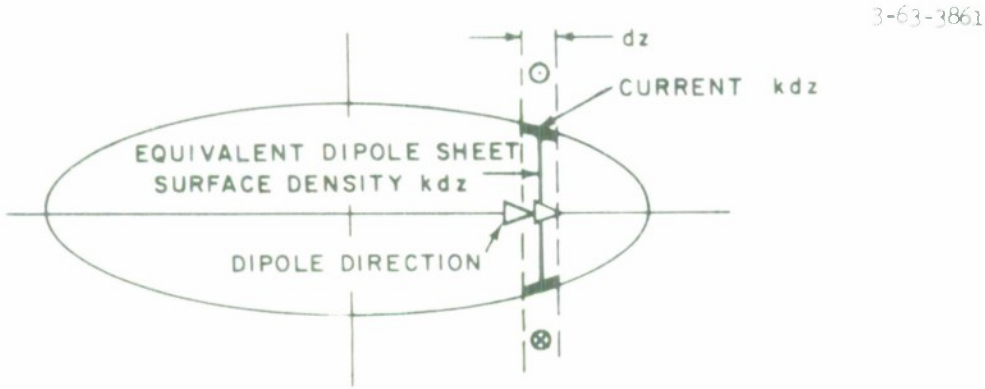

Figure A-5 Dipole Sheet Construction for Determination of External Field

A similar approach can be used to find the field inside the ellipsoid. In this case the dipole sheets must not penetrate the interior. Figure A-6 shows the appropriate construction.

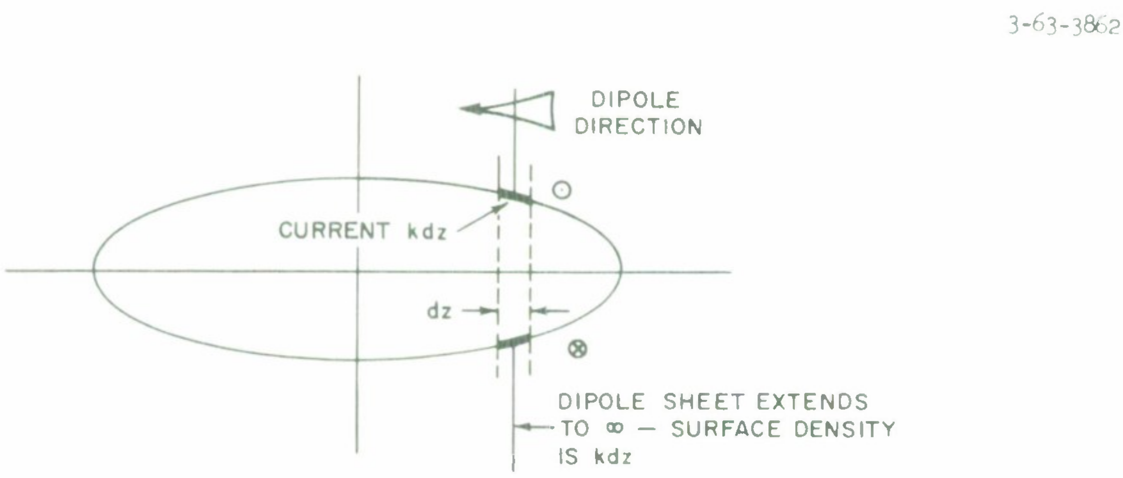

Figure A-6 Dipole Sheet Construction for Determination of Internal Field

Thus the internal field is that produced by an infinite slab of uniformly magnetized material with an ellipsoid hole in it. As shown in Figure A-7, this field is easily determined by superposition

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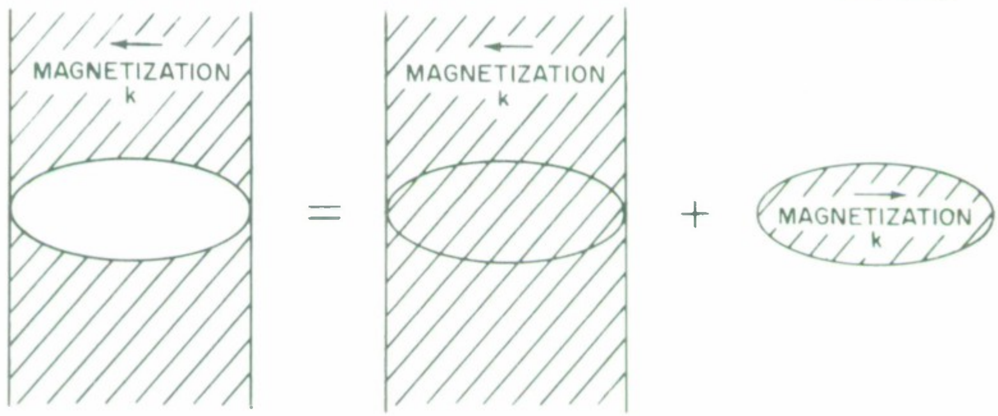

Figure A-7 Determination of Internal Field by Superposition

The slab produces an H field in the plus z direction, since it behaves like a parallel plate capacitor with surface charge density $\mu_o k$. The magnitude of this field is k. The ellipsoid produces a field of β k in the negative z direction. The total internal field is thus $(1-\beta)$ k in the plus z direction.

3. Determination of Fields in a Wound, Magnetizable, Prolate Ellipsoid

The total H field in the plus z direction is:

$$H = (1 - \beta) k - \beta M$$

Since:

$$M = \frac{B}{\mu_o} - H$$

we see

$$H = \frac{\beta}{1 - \beta} \quad \frac{B}{\mu_o} = k$$

This leads to the load line construction shown in Figure A-8

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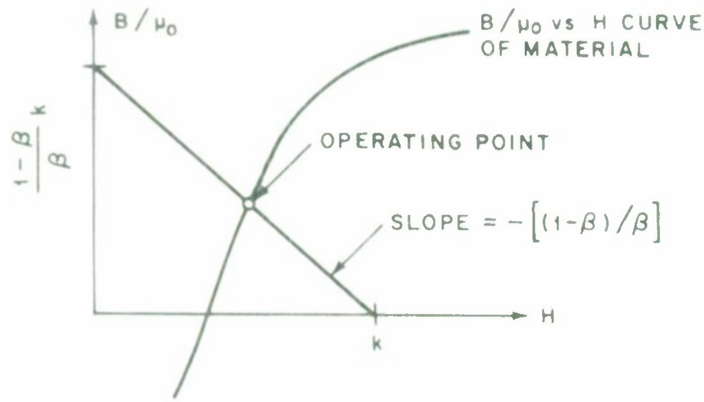

Figure A-8 Load Line Construction

4. Relation to Cylindric Rod

Figure A-9 shows a cylindric rod and an approximating ellipsoid. If we select the dimensions of the ellipsoid so that:

$$\frac{b}{r} = \frac{a}{\ell/2} = \left(\frac{3}{2}\right)^{1/3} \simeq 1.114$$

then the two volumes will be equal.

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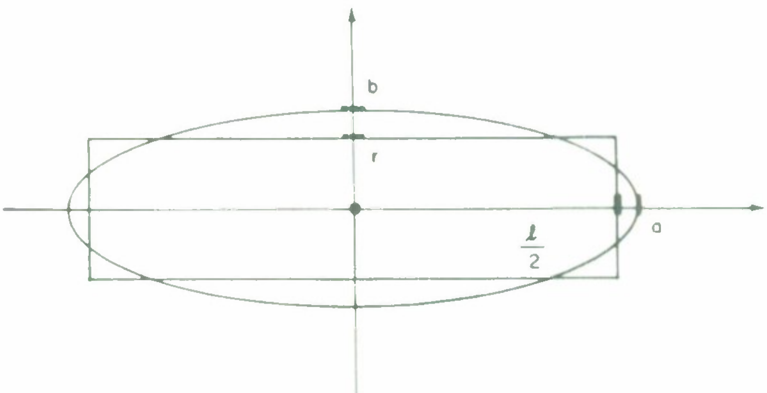

Figure A-9: Fitting an Ellipsoid to a Cylinder

The fit is pretty good - the ellipsoid protrudes only about 11 per cent outside the cylinder.

To approximate to the factor f employed in the body of the paper we equate the slopes of the two load lines in Figures 2 and A-8

$$f \frac{CV}{\ell^3} = \frac{\beta}{1-\beta}$$

$$f = \frac{\beta}{1-\beta} \cdot \frac{\ell^2}{\pi r^2} = \frac{\beta}{1-\beta} \cdot \frac{4\Lambda^2}{\pi}$$

For large Λ we may approximate:

$$\beta \simeq \frac{\text{Log } 2\Lambda - 1}{\Lambda^2}$$

so:

$$f \simeq \frac{4}{\pi} \frac{\text{Log } 2\Lambda - 1}{1 - \frac{\text{Log } 2\Lambda}{\Lambda^2}}$$

A typical rod with a length of 50 cm and diameter of .5 cm has a Λ of 100 and an f of approximately 5.5.

APPENDIX B

Air Core Loop Relations

The following formulae are useful for comparison purposes. In all cases we have been concerned with the loop has consumed enormously more power than the rod with a ferromagnetic core. Nonetheless, the formulae below are provided to facilitate comparisons in cases with very different limitations.

$$\mathfrak{M} = NiA = \pi Ni r^2$$

where r is the radius of the loop. Since the mean turn length is $2\pi r$, the wire area is $WV/2\pi Nr$. Hence the resistance is:

$$R_{coil} = \frac{\rho}{WV}(2\pi Nr)^2$$

From these one derives the hold power:

$$P_{hold} = \frac{\rho}{WV}\left(\frac{2\mathfrak{M}}{r}\right)^2$$

Clearly one should use as large an r as possible. Moreover, if weight is fixed, one wants a material with a low ρd product - i.e., aluminum.

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13. ABSTRACT

This report is a compilation of considerations involved in the design of electromagnetic rods for satellite erection. Formulas are derived showing the effect of various choices of core and wire material and weight, number of cores, length of cores, number of turns per core, and wire area on the magnetic moment, hysteresis power, hold power, hold voltage, and switching time. These formulas are then applied to the problem of producing a required magnetic moment at the least cost in weight and power consumption.

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